How a Bicycle is Propelled, Generally

Briefly, a bicycle is a vehicle with two wheels made for getting yourself around. On almost all bicycles, the rider pushes on the pedals with his or her legs to cause the rear wheel to turn and propel the bike forward. These pedals are attached to pedal arms or “cranks” which are in turn attached to a “chainring”. Every time the pedals and cranks go around, the chainring goes around. This chainring then pulls a chain which in turn pulls to a “cog”. This rear cog causes the rear wheel to spin and the bike to zip away. Please see the Figure 1.

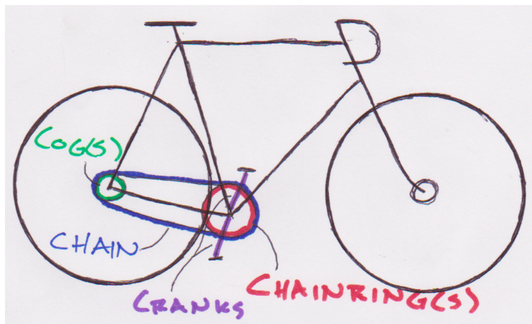

Figure 1: Feet push the pedals, pedals turn the cranks, cranks turn the chainring, chainring pulls the chain, chain turns the cogs, cogs turn the back wheel, and away you go.

Bicycle Gears

The chainring toward the front of the bike and the cog toward the back of the bike each have a particular number of teeth that mesh with the chain. These numbers are proportional to the diameters of the chainrings and cogs. In order to increase the speed with which one can move, the size of the chainring attached to the cranks is generally much larger than the cog attached to the rear wheel. For instance, a typical chainring might have 39 teeth while the cog might have only 13. Such a bike could be said to have a “gear ratio” of 39:13 or 3:1. This setup will cause the cog and rear wheel to turn three times each time the chainring and cranks turn once. One can perform all kinds of calculations using the formulas of the circumference of a circle to find the distance travelled if you have such-and-such a ratio and turn the cranks so-and-so many times. For reference, wheels and tires are usually about 27 inches or 700 mm in diameter.

As you might imagine, as the gear ratio rises, the effort necessary to propel the bike increases. For instance, if you were riding with a 52:13 or 4:1 gear ratio, the rear wheel would turn four times every time you turn the cranks around. You could go super fast, but you’d have to push the pedals really hard. Conversely, if you were riding with a 26:13 or 2:1 gear ratio, you probably wouldn’t be going that fast, but it’d be super easy to pedal. Generally one uses high gear ratios when going downhill or racing and low gear ratios when doing things like going up hills, hauling a heavy load, or taking it really easy.

Standard Bike vs Single-Speed vs Fixed-Gear Bike vs Coaster Brake

In order to give bikes the most versatility, they are usually equipped with "derailers" that allow a rider to change gears while riding. These derailleurs allow the rider to select from a number of different gear ratios as the terrain or situation changes. For instance, on my bike I can select from a 39- or 52-tooth chainring with my front derailler and cogs ranging from 12 to 28 teeth with the rear derailler. This allows me to select ratios ranging from 39:28 $\approx$ 1.4:1 to 52:12 $\approx$ 4.3:1. So, every time I turn my cranks once, my rear wheel will turn between about 1.4 and 4.3 times, depending on which gear I have selected.

In the past decade or so, bikes with only one chainring and one cog have become much more popular. These are called "single-speed" bikes[1]. With this kind of bike, there are no deraillers and only one gear ratio is available. These lack the versatility that changing gears on a standard bike allows, but they can be nice, especially if you live out in Hollywood or some other area with not too many hills. These kinds of bikes are very easy to maintain, require less mental energy to ride, and are generally very light and pretty-looking.

Besides single-speed bikes, there is another kind of bicycle that has become popular in recent times. This kind of bike is a called a "fixed-gear" or "fixie" bike. On a standard bike, when you stop pedalling, a racheting mechanism between the cogs and the rear wheel allow the rear wheel to continue spinning even when there is no pedalling and the cranks aren't turning. This moving without pedaling is called "coasting" and is pretty great. Fixies remove the ability to coast by forgoing the ratcheting system between the cogs and rear wheel. So, the cranks and chainring and rear cog and rear wheel are permanently coupled. On this sort of bike, whenever the bike is moving forward and the rear wheel is spinning, the rear cog and thereby the chain, chainring, cranks, and pedals are also spinning around. So, if you want your bike to keep moving, you had better keep pedalling. Conversely, if your bike is still going forward, the cranks and your pedals are going to be moving too.[2] Interestingly, this allows you to stop your bike by pushing backwards on the pedals and resisting the drivetrain's movement forward.

These fixie bikes are all the rage for a bunch of teenagers these days, despite the fact that they're harder to ride. Reasons for this may include that they're sleek looking, they're simpler, they're easy to maintain, they're in fashion, they can be relatively cheap, they have a higher learning curve, they're easy to do tricks on, they let you easily maintain a constant speed on group rides, and they're fun.[3]

There is a fourth sort of bicycle that is called a "coaster brake" bike that has only one gear ratio and allows for coasting yet also has a mechanism built into the rear hub that allows the rider to lock up the rear wheel by pushing backwards on the pedals. This kind

[1] makes sense

[2] You gotta be careful lest you forget to keep moving your legs and the pedals throw you off of the bike.

[3] SAFETY NOTE: Lots of folks like to ride "brakeless" fixies because they're what the hardcore racers ride and they are, admittedly, sleeker and cooler looking. However, coolness notwithstanding, I strongly recommend putting a front brake on all bicycles for two reasons. Firstly, though you can stop these bikes by resisting the forward motion of the pedals, it seems wise to me to have two ways to stop your bike in case one of them breaks down unexpectedly. (This happens more often than you might expect. And putting your shoe directly on the wheel doesn't really count.) Secondly, it is not possible to stop your bike as quickly with the rear wheel as you can with the front wheel because when you are deccellerating, most of your weight ends up on the front wheel and the stopping force for your bike is proportional to this weight the wheel being stopped. It is your front brake that is going to save you from ending up under a bus in dangerous situations.

of braking system is often employed on older bikes, beach crusiers, and little kids' bikes.

Skid Patches

What I really would like to talk about is a math-related phenomenon that occurs with fixed-gear bicycles. As stated earlier, because fixed-gear bicycles don't have the ratchet mechanisms to allow coasting, you can stop them by simply pushing backwards on the pedals. If you do a bit of a trick and push backwards on the pedals hard enough, you can cause the rear wheel and the cranks to lock-up and send the bike into a skid-stop. During a skid-stop, the rear wheel doesn't turn at all and the tire skids across the pavement. Even though it doesn't usually stop you so fast, this is a legitimate and fun way to stop your bike.

However, because the tire is skidding, this method of stopping can cause your tires to wear through rather quickly. As the tire skids across the pavement, the pavement wears through the rubber and causes "skid patches," or areas of extreme skid-caused wear, to form. If these skid patches become severe enough, they will eventually wear a hole through your tire and cause the inner tube to pop and the tire to deflate, sometimes with disasterous consequences. This may particularly be a problem if one is unlucky enough to constantly skid on the same section of tire and create only very few, very common skid patches. *Luckily, using mathematics we may pick a gear ratio for fixed-gear bicycles that results in a large number skid patches over a much larger area of the tire than we might otherwise have!* Though these will not prevent skid patches from forming, they will spread the wear out over a much greater area of the tire and extend the tire's life.[4]

Again, because the ratcheting mechanism on a fixed-gear bicycle is not present, whenever the bicycle is moving and the rear wheel is moving, so too is the entirety of the drivetrain. This drivechain includes cog, the chain, the chainring, the cranks, and the pedals. The reverse is also true; whenever the drivetrain / pedals / cranks / chainrings / chain / cog is moving, so is the rear wheel. Moreover, all of these parts are *always* moving proportionally. For instance, if you have a 3:1 gear ratio, whenever you turn the cranks once, the rear wheel will turn exactly three times, not the three-or-more times that turning the cranks once would yield on a standard bicycle equipped with a ratcheting mechanism for coasting.

Now, when one stops the bicycle with an aforementioned skid-stop, the braking position of the cranks and chainring is not arbitrary. In fact, because once has to apply a fair amount of force on the pedals and cranks to resist the force that braking applies through the drivetrain, the rider usually causes the cranks and chainring to stop in the same relative position[5] every time he or she does a skid-stop. It's just easier on the knees/legs. So, the number of turns of the cranks and crainring between each skid-stop is always going to be an integer or pretty close to an integer. This becomes a problem if one has a gear ratio of, say, 3:1; because the rear wheel will turn exactly three times for each time the cranks go around and there is an integer number of crank rotations between every skid-stop, the number of rear-wheel turns between each skid-stop will also be an integer. This will cause the rear wheel to stop in the same relative position every time you brake with a skid-stop and cause

[4]You can also spread the skid patches around by removing your rear-wheel and re-setting it relative to your chainring, but this is an inelegant solution and a huge hassle. You could also just not skid-stop so often to save your tires, I guess. This seems unlikely though.

[5]When I say "relative position," I just mean the part of the rotation left after you discard the integer part of the rotation. For instance, if the cranks turn around 2.25 times, I'd say the relative position is .25. It's kind of like a remainder.

you to have exactly one skid patch. In this case, the skid patch will be very severe and ruin your tires post-haste.

Number of Skid Patches and Gear Ratio Denominators

So then, how can we create a larger number of much less severe skid patches? We should like to give the rear wheel the greatest number of relative positions available when the rider is performing a skid-stop. Let's consider a gear ratio of 52:17 or 3 1/17:1. Every time the cranks turn around once, the rear wheel will turn around three and one-seventeenth times. So, for turning the cranks around once, we'll get 3 1/17 rear wheel turns, for twice, we'll get 6 2/17 turns, for three times, we'll get 9 3/17 turns, etc. This will result in the relative positions 0/17, 1/17, 2/17, 3/17, ..., 15/17, and 16/17 being available when the rider is performing a skid-stop.[6] This means that the rider will have seventeen different skid-patches on his or her tire and the tire will likely last much longer than it would otherwise with a ratio of, say, 51:17 or 3:1.

But why did 52:17 result in so many skid patches? What if I want to figure out how many skid patches I'll have with some other gear ratio? *It turns out that the number of skid patches one will have is the same as the denominator when you write the gear ratio as a fraction in lowest terms!* But why?

Let's make an argument. First we should find out how many full rotations of the cranks it takes until we get the wheel to the relative position we started with. That is, how many full rotations of the cranks must we have before the rear wheel turns around and integer number of times and we're back where we started, relative rotationally speaking? This will be the maximum number of different skid patches we can have. Let's let k be the number of times that the cranks have turned and p/q be the gear ratio where p/q is written in lowest terms. In that case, the number of times that the rear wheel turns around is

$$k\frac{p}{q}.$$

What we are concerned about is the number k of full rotations of the cranks result in the rear wheel being in the same relative position we started with. At this point, the wheel will have spun around exactly some integer number of times. Let's call this integer n. So, we are trying to find the smallest number k such that

$$k\frac{p}{q} = n$$

or

$$kp = qn$$

where n is any integer.

Because n, p, q, and k are all integers, for $kp = qn$, we need the products on both sides of the equation to have the same factors. Because our gear ratio p/q was written in lowest terms, we know that p and q do not have any common factors. So, we need our number of crank rotations k to have at least all the factors contained in the denominator of our gear ratio, q. So, $k \geq q$. But also, $k \leq q$ because if we simply let $k = q$ and $n = p$, we have $kp = qn$. So, we can let $k = q$ and have q full crank rotations before we end up at the same

[6] This is assuming that the rider isn't doing something really silly like only skid-stopping after the turning the cranks around a multiple of seventeen times. Shouldn't have to worry about that.

relative position we started in. Then, we'll get all the same relative positions over again. So, with a p/q gear ratio with p/q in lowest terms, we'll get at most q different relative positions.

Now, we should show that we can get *at least* q different relative positions with a p/q gear ratio, where p/q is in lowest terms. That is, let's show that each of the intermediate full crank rotations puts the rear wheel at a different relative position. Again, p/q is in lowest terms so p and q have no common factors. Let's assume that when we turn the cranks l times and when we turn the cranks m times, the rear wheel ends up in the same relative position, where $l, m < q$. This would mean that lp/q and mp/q have the same remainder. In other words, $lp \equiv mp \bmod q$. Because p and q don't have any common factors, this means that $l \equiv m \bmod q$. And because $l, m < q$, this means that $l = m$. So, the only way that we get a particular relative position by turning the cranks fewer than q times is by turning the cranks some unique particular amount of times. This means that turning the cranks each of $0, 1, 2, 3, \ldots, q-1$ times yeilds a different relative position. So we end up with at least q different skid patches where q was the denominator of the gear ratio when written in lowest terms. So, the moral of the story is that with a gear ratio of p/q (in lowest terms), we can get exactly q different skid patches.

In practical, prescriptive-type terms, this means that it's wise to pick your cog and chainring to set up a gear ratio with a large denominator. Some simple strategies for this might be picking a chainring with 37, 41, 43, 47, or 53 teeth (these are all primes and will almost certainly result in a gear ratio with a denominator equal to the number of teeth on the cog) or picking a $16 = 2^4$-tooth cog and a chainring with an odd number of teeth (this will result in a gear ratio with a denominator of 16). Such a move will keep your tires lasting longer and save you expenditure of money, time, and effort.